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Partial Hyperbolicity, Lyapunov Exponents and Stable Ergodicity

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We present some results and open problems about stable ergodicity of partially hyperbolic diffeomorphisms with non-zero Lyapunov exponents. The main tool is local ergodicity theory for non-uniformly hyperbolic systems.

KEY WORDS: Partial hyperbolicity; Lyapunov exponents; accessibility; stable ergodicity.

1. INTRODUCTION

1.1. Consider a C^2 diffeomorphism f of a compact smooth Riemannian manifold M preserving a smooth probability measure μ. Assume that f has nonzero Lyapunov exponents on an invariant set $\mathscr{L}$ of positive measure. It is well-known (see ref. 1) that the ergodic components of $f \mid \mathscr{L}$ are all of positive measure, and hence there can be only contably many such components.

However, not much is known about the topological structure of the set $\mathscr{L}$ nor about the topological structure of ergodic components. In particular, one may wonder whether ergodic components (and hence the set $\mathscr{L}$) are open (mod 0)—the phenomenon known as local ergodicity. Some criteria for local ergodicity were obtained in refs. 1 and 2 while some basic ideas go back to pioneering work of Ruelle[(3)] and Sinai.[(4)] In ref. 5, the

Dedicated to the great dynamicists David Ruelle and Yakov Sinai on their 65th birthdays.

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authors constructed an example of a volume preserving diffeomorphism with nonzero Lyapunov exponents and a countable (not finite) number of ergodic components which are all open (mod 0). Note that if the ergodic components are open (mod 0) and f is topologically transitive then $f\,|\,\mathscr{L}$ is ergodic.

In this connection the following two problems are of interest:

Problem 1. Is there a volume preserving diffeomorphism with a.e. nonzero Lyapunov exponents such that some (or even all) of the ergodic components with positive measure are not open (mod 0)?

Problem 2. Is there a volume preserving diffeomorphism which has nonzero Lyapunov exponents on an open (mod 0) and dense set U which has positive but not full measure? Is there a volume preserving diffeomorphism with the above property such that $f\,|\,U$ is ergodic?

1.2. Systems with nonzero Lyapunov exponents are nonuniformly hyperbolic. In this paper we deal with the situation when hyperbolicity is uniform throughout the manifold in some but not all directions. More precisely, we assume that f is partially hyperbolic, i.e., the tangent bundle TM can be split into three df-invariant continuous subbundles (distributions)

$$TM = E^s \oplus E^c \oplus E^u.$$

The differential df contracts uniformly over $x \in M$ along the *strongly stable* subspace $E^s(x)$, it expands uniformly along the *strongly unstable* subspace $E^u(x)$, and it can act either as nonuniform contraction or expansion with weaker rates along the *central* direction $E^c(x)$; see the next section for more details. Partially hyperbolic systems were first studied in the 1970's by Brin, Pesin, Hirsch, Pugh and Shub, see, e.g., refs. 6–9. The definition given here was introduced in ref. 6. The presence of uniformly contracting and expanding directions is crucial in studying the global behavior and ergodic properties of these systems.

1.3. The distributions $E^s(x)$ and $E^u(x)$ are integrable and their integrable manifolds form two transversal foliations of M, the *strongly stable* and *strongly unstable foliations* of M, which we denote by W^s and W^u respectively. For every $x \in M$ the leaves $W^s(x)$ and $W^u(x)$ of the foliations containing x are smooth immersed submanifolds in M called the *strongly stable* and *strongly unstable global manifolds* at x. If $y \in W^s(x)$, then

$d(f^n(x), f^n(y)) \to 0$ with an exponential rate as $n \to \infty$, and if $y \in W^u(x)$, then $d(f^n(x), f^n(y)) \to 0$ with an exponential rate as $n \to -\infty$.

1.4. We say that a partially hyperbolic diffeomorphism with invariant measure μ has *negative central exponents* (on a set A) if for μ-a.e., x (in the set A) we have $\chi(x, v) < 0$ for all nonzero $v \in E^c(v)$, where $\chi(x, v)$ is the Lyapunov exponent (defined in the next section). The definition of *positive central exponents* is analogous. When f has negative central exponents on A, the strongly unstable subspace $E^u(x)$ includes all of the expanding directions at x for a.e. $x \in A$.

By exploiting continuity and the absolute continuity property of the strongly unstable foliation one can establish the following result.

Theorem 1. Let f be a C^2 diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that there exists an invariant subset $A \subset M$ with $\mu(A) > 0$ on which f has negative central exponents. Then every ergodic component of $f \mid A$ is open (mod 0) and so is the set A.

If the map f is topologically transitive, then A is dense and $f \mid A$ is ergodic.

1.5. Apparently topological transitivity does not guarantee that the set A is of full measure and one needs a stronger requirement, which we now discuss.

Two points $p, q \in M$ are called *accessible* if there are points $p = z_0, z_1, \ldots, z_{\ell-1}, z_\ell = q$, $z_i \in M$ such that $z_i \in W^u(z_{i-1})$ or $z_i \in W^s(z_{i-1})$ for $i = 1, \ldots, \ell$. The collection of points $z_0, z_1, \ldots, z_\ell$ is called a *us-path* connecting p and q and is denoted by $[p, q] = [z_0, z_1, \ldots, z_\ell]$. Accessibility is an equivalence relation. The diffeomorphism f is said to have the *accessibility property* if the partition into accessibility classes is trivial (i.e., any two points $p, q \in M$ are accessible) and to have the *essential accessibility property* if the partition into accessibility classes is ergodic (i.e., a measurable union of equivalence classes must have zero or full measure).

It was shown in ref. 10 that if a.e. pair of points in M is joined by a *us*-path, then the orbit of, a.e., point is dense in M. We shall see that the proof actually requires only essential accessibility. This and Theorem 1 imply the following result.

Theorem 2. Let f be a C^2 partially hyperbolic diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that f has negative central exponents on an invariant set A of positive measure and is essentially accessible. Then f has negative central exponents on the whole of M, the set A has full measure, f has nonzero Lyapunov exponents, a.e., and f is ergodic.

1.6. The proofs of Theorems 1 and 2 are based on methods developed in refs. 1 and 11. In proving Theorem 2 we also use some ideas from ref. 12 to make a crucial step: once a partially hyperbolic diffeomorphism has negative Lyapunov exponents in the central direction on a set of positive measure, then this set indeed has full measure. This phenomenon has been observed in other situations, for example billiards and geodesic flows on negatively curved manifolds.

1.7. Accessibility plays a crucial role in stable ergodicity theory. A diffeomorphism f is called *stably ergodic* if it preserves a smooth measure μ and there exists a C^2-open neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^2(M)$ such that any diffeomorphism $g \in \mathcal{U}$ which preserves μ is ergodic with respect to μ. Volume preserving Anosov diffeomorphisms are stably ergodic. Recently Grayson *et al.*[13] proved that the time one map of the geodesic flow of a surface of constant negative curvature is stably ergodic. This result has been generalized several times.[12, 14–16] These papers give conditions under which a partially hyperbolic diffeomorphism f is stably ergodic (with respect to a smooth measure on M). Among these conditions, the most crucial one is stable accessibility. A diffeomorphism f is said to be *stably accessible* if there exists a C^1-open neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^2(M)$ such that any diffeomorphism $g \in \mathcal{U}$ is accessible. Pugh and Shub[18] have formulated two conjectures relating accessibility and stable ergodicity.

Conjecture 1. Stably ergodic diffeomorphisms form an open and dense set in the space of partially hyperbolic diffeomorphisms.

Conjecture 2. A volume preserving partially hyperbolic diffeomorphism with the essential accessibility property is ergodic. In particular a volume preserving partially hyperbolic diffeomorphism that is stably accessible should be stably ergodic.

In ref. 16, Pugh and Shub proved Conjecture 2 under some additional assumptions of which the most restrictive one is *center-bunching,* that is the norm $\|df^{\pm 1} \mid E^c(x)\|$ should be close to 1 uniformly over x. A natural way to relax this condition is to consider its nonuniform version. That is, to consider the cases in which the Lyapunov exponents in the central direction are: (1) all negative (or all positive), (2) all nonzero (i.e., some negative and some positive), (3) all zero, or (4) not all nonzero (i.e., some zero). We think that splitting the study of stable ergodicity of partially hyperbolic diffeomorphisms into these four cases may be rewarding. In fact, the approach of Pugh and Shub gives no information on the quantitative properties of the diffeomorphisms they consider and quite different tools are currently used to obtain some quantitative information on the system in the non-uniformly hyperbolic and zero exponent cases (see refs. 19 and 20).

In this paper we study the first of the above cases. Surprisingly, the diffeomorphisms we consider are stably ergodic under the much weaker assumption that only the unperturbed map is accessible. In other words no information about the perturbation is required to establish stable ergodicity.

Theorem 3. Let f be a C^2 partially hyperbolic diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that f is accessible and has negative central exponents on a set of positive measure. Then f is stably ergodic.

We also prove a related result.

Theorem 4. Let f be a C^2 partially hyperbolic diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that f is accessible and

$$\int_M \ln \|df \mid E^c_f(x)\| \, d\mu(x) < 0.$$

Then f is stably ergodic.

Ideas from ref. 21 about hyperbolic times are used in the proof of these theorems.

1.8. We now consider the case when f is partially hyperbolic and has positive central exponents. Note that the inverse map is partially hyperbolic, preserves the measure μ, and has negative central exponents. Applying the earlier results to the inverse map we obtain that Theorems 1, 2, and 3 hold for f. Theorem 4 also holds if the inequality is replaced by

$$\int_M \ln \|df^{-1} \mid E^c_f(x)\|^{-1} \, d\mu(x) > 0.$$

1.9. In view of Theorem 3 the following problems are of interest.

Problem 3. Is Conjecture 2 true if all Lyapunov exponents of f are non-zero?

Problem 4. Is Conjecture 2 true if all Lyapunov exponents of f in the central direction are zero?

1.10. An important example of a diffeomorphism satisfying the condition of Theorem 3 was constructed in ref. 22. It is a small perturbation of a circle extension of an Anosov diffeomorphism. Further examples, including modifications of Anosov diffeomorphisms and time-one maps of Anosov flows can be found in ref. 12 and 19. In this connection we propose the following problem.

Problem 5. Let f be a stably ergodic diffeomorphism. Can it be approximated by a diffeomorphism having (stably) non-zero Lyapunov exponents?

See refs. 23 and 24 for some evidence that this might be true.

1.11. Some of the methods presented here were used in ref. 10 to show that any manifold carries a Bernoulli diffeomorphism with non-zero Lyapunov exponents. The following problem arises naturally.

Problem 6. Which manifolds carry an open set of C^k diffeomorphisms with non-zero Lyapunov exponents?

For $k = 1$, it follows from a result of Mane[23, 25] that $\mathbb{T}^2$ is the only surface with this property. Thus the answer to this question is not always positive. It seems likely (see Problem 5) that the answer is positive if the manifold admits a partially hyperbolic diffeomorphism, but even this is unknown.

2. PRELIMINARIES

See refs. 26–28 for more details.

A diffeomorphism f of a compact smooth Riemannian manifold M is called *(uniformly) partially hyperbolic* if for every $x \in M$ the tangent space at x admits an invariant splitting

$$T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$$

into *strongly stable* $E^s(x) = E^s_f(x)$, *central* $E^c(x) = E^c_f(x)$, and *strongly unstable* $E^u(x) = E^u_f(x)$ subspaces. This means that there exist numbers

$$0 < \lambda_s < \lambda'_c \leqslant 1 \leqslant \lambda''_c < \lambda_u$$

such that for every $x \in M$,

$$v \in E^s(x) \Rightarrow \|d_x f(v)\| \leqslant \lambda_s \|v\|,$$
$$v \in E^c(x) \Rightarrow \lambda'_c \|v\| \leqslant \|d_x f(v)\| \leqslant \lambda''_c \|v\|,$$
$$v \in E^u(x) \Rightarrow \lambda_u \|v\| \leqslant \|d_x f(v)\|.$$

Given $x \in M$, one can construct *strongly stable* and *strongly unstable local manifolds* at x. We denote them by $V^s(x)$ and $V^u(x)$ respectively. They can be characterized as follows: there is a neighborhood $U(x)$ of the point x such that

$$V^u(x) = \{y \in U(x) : d(f^{-n}(x), f^{-n}(y)) \leqslant C\lambda_u^{-n}\, d(x, y) \text{ for all } n \geqslant 0\},$$

and

$$V^s(x) = \{y \in U(x) : d(f^n(x), f^n(y) \leqslant C\lambda_s^n\, d(x, y) \text{ for all } n \geqslant 0\}.$$

Let us stress that the sizes of the strongly stable and strongly unstable local manifolds are uniformly bounded from below.

We define the *strongly stable* and *strongly unstable global manifolds* at x by

$$W^u(x) = \bigcup_{n \geqslant 0} f^n(V^u(f^{-n}(x))),$$
$$W^s(x) = \bigcup_{n \geqslant 0} f^{-n}(V^s(f^n(x))).$$

Recall that a partition ξ of M is called a *foliation* if there exist $\delta > 0$, $q > 0$, and an integer $k > 0$ such that for each $x \in M$:

(1) There exists a smooth immersed k-dimensional manifold $W(x)$ containing x for which $\xi(x) = W(x)$ where $\xi(x)$ is the element of the partition ξ containing x. (The manifold $W(x)$ is called the *(global) leaf* of the foliation at x; the connected component of the intersection $W(x) \cap B(x, \delta)$ that contains x is called the *local leaf* at x and is denoted by $V(x)$; the number δ is called *the size* of $V(x)$.)

(2) There exists a continuous map $\phi_x \colon \mathscr{L} \cap B(x, q) \to C^1(D, M)$ (where D is the unit ball) such that $V(y)$ is the image of the map $\phi_x(y) \colon D \to M$ for each $y \in B(x, q)$.

The strongly stable and strongly unstable global manifolds form two transversal foliations of M.

We denote by

$$\chi(x, v) = \limsup_{n \to \infty} \frac{1}{n} \log \|df^n v\|$$

the *Lyapunov exponent* of a nonzero vector v at $x \in M$ and by $\chi^i_f(x)$ the values of the Lyapunov exponents at x. Note that the functions $\chi^i_f(x)$ are invariant. There exists a subset $\Lambda \subset M$ of full measure which consists of *Lyapunov regular points* (see ref. 27, Sections 1.5 and 2.1). Among other things Lyapunov regularity of x means that

$$\chi(x, v) = \lim_{n \to \pm\infty} \frac{1}{n} \log \|df^n v\|$$

for all nonzero $v \in T_x M$.

An invariant measure μ is called *hyperbolic* on a set $\mathscr{L}$ if $\mu(\mathscr{L}) > 0$ and μ-a.e., $x \in \mathscr{L}$ has the property that $\chi(x, v) \neq 0$ for all nonzero $v \in T_x M$. By neglecting a set of measure 0 we may assume that $\mathscr{L} \subset \Lambda$. For every $x \in \mathscr{L}$ the tangent space at x admits an invariant splitting

$$T_x M = E^-_f(x) \oplus E^+_f(x)$$

into *stable* and *unstable* subspaces. Let $\lambda_-(x) = e^{\chi_-(x)}$ and $\lambda_+(x) = e^{\chi_+(x)}$ where $\chi_-(x)$ and $\chi_+(x)$ are respectively the largest negative Lyapunov exponent and the smallest positive Lyapunov exponent at x. For each $\varepsilon > 0$ there are Borel functions $C(x) > 0$ and $K(x) > 0$ such that

(1) for each $n > 0$,

$$\|df^n v\| \leqslant C(x)\, \lambda_-(x)^n\, e^{\varepsilon n} \|v\|, \qquad v \in E^-(x),$$

$$\|df^{-n} v\| \leqslant C(x)\, \lambda_+(x)^{-n}\, e^{-\varepsilon n} \|v\|, \qquad v \in E^+(x);$$

(2) the angle

$$\angle\,(E^-(x), E^+(x)) \geqslant K(x);$$

(3) for each $m \in \mathbb{Z}$,

$$C(f^m(x)) \leqslant C(x)\, e^{\varepsilon |m|}, \qquad K(f^m(x)) \geqslant K(x)\, e^{-\varepsilon |m|}.$$

For every $x \in \mathscr{L}$ one can construct *stable* and *unstable local* manifolds $V^-(x)$ and $V^+(x)$. They can be characterized as follows: there is a neighborhood $U(x)$ of the point x such that $V^+(x)$ is the set of all $y \in U(x)$ for which

$$d(f^{-n}(x), f^{-n}(y)) \leqslant C(x)\, \lambda_+(x)^{-n}\, e^{-\varepsilon n}\, d(x, y) \qquad \text{for all} \quad n \geqslant 0,$$

while $V^-(x)$ is the set of all $y \in U(x)$ for which

$$d(f^n(x), f^n(y) \leqslant C(x)\, \lambda_-(x)^n\, e^{\varepsilon n}\, d(x, y) \qquad \text{for all} \quad n \geqslant 0.$$

The manifolds $V^-(x)$ and $V^+(x)$ are tangent at x to $E^-(x)$ and $E^+(x)$ respectively. If f is partially hyperbolic, $V^+(x) \supset V^u(x)$ and $V^-(x) \supset V^s(x)$.

The sizes of the stable and unstable local manifolds vary with x in a measurable way. They are not always uniformly bounded from below, in contrast to the sizes of the strongly stable and strongly unstable local manifolds. If $\delta(x)$ is the size of a stable or unstable local manifold at x, then for every $m \in \mathbb{Z}$

$$\delta(f^m(x)) \geqslant \delta(x)\, e^{-\varepsilon\, |m|}.$$

It is known that the function $\delta(x)$ depends only on $C(x)$ and $K(x)$; in particular, $\delta(x)$ is uniformly bounded from below if $C(x)$ is uniformly bounded from above and $K(x)$ is uniformly bounded from below.

The families of these local manifolds possess the *absolute continuity property*. This means the following. Denote by $m^u(x)$ the Riemannian volume on $V^u(x)$ induced by the Riemannian metric on $V^u(x)$ as a smooth submanifold in M. Given $x \in M$ and sufficiently small $r > 0$, consider the partition ξ^u of $B(x, r)$ (the ball centered at x of radius r) by strongly unstable local manifolds $V^u(y)$ with $y \in B(x, r)$. Let μ^u be the conditional measure generated by μ on $V^u(y)$, $y \in B(x, r)$. Then the measures $m^u(y)$ and $\mu^u(y)$ are equivalent for, a.e., $y \in B(x, r)$.

The families of local manifolds $V^s(x)$, $V^+(x)$ and $V^-(x)$ also possess the absolute continuity property.

In this paper we deal with the case where f is partially hyperbolic and has negative central Lyapunov exponents on a set of points x of positive or full measure (with respect to an invariant measure μ on M). For such x we have $E^-(x) = E^s(x) \oplus E^c(x)$. In particular, $V^+(x) = V^u(x)$. In this case, we will use the notation $V^{cs}(x)$ for the stable local manifold $V^-(x)$ and we define the stable global manifold

$$W^{cs}(x) = \bigcup_{n \geqslant 0} f^{-n}(V^{cs}(f^n(x))).$$

3. PROOFS

3.1. Proof of Theorem 1

Let us call a point z *Birkhoff regular* if the Birkhoff averages

$$\varphi^-(z) = \lim_{n\to-\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(z)) \qquad \text{and} \qquad \varphi^+(z) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(z))$$

are defined and equal for every continuous function φ on M. Applying Birkhoff's ergodic theorem to a countable dense subset of the continuous functions shows that the set $\mathscr{B}$ of Birkhoff regular points has full measure in M with respect to μ. It follows from the absolute continuity of the stable local manifolds $V^-(x)$ that μ-a.e. $x \in A$ is Lyapunov regular and has the property that m^--a.e. $z \in V^-(x)$ belongs to $\mathscr{B}$, where m^- is the Riemannian volume on V^-.

We shall show that any point $x \in A$ with the above properties has a neighborhood in which the backwards Birkhoff average φ^- is a.e. constant for any continuous function φ. Since f has negative central exponents at x, we have $V^-(x) = V^{cs}(x)$. The disc $V^{cs}(x)$ is transverse to the strong unstable foliation. This and the uniform size of the strongly unstable local manifolds $V^u(z)$ ensure that the set

$$N(x) = \bigcup_{z\in V^{cs}(x)} V^u(z)$$

is a neighbourhood of x. The full measure of $\mathscr{B}$ in $V^{cs}(x) = V^-(x)$ and the absolute continuity of the strongly unstable local manifolds ensure that

$$N'(x) = \bigcup_{z\in V^{cs}(x)\cap\mathscr{B}} V^u(z)\cap\mathscr{B}$$

has full measure in $N(x)$.

We now use the Hopf argument to show that φ^- is constant on $N'(x)$ for any continuous function φ. Let $y \in N'(x)$. Then $y \in V^u(z)$ for a point $z \in V^-(x) \cap \mathscr{B}$. Since backwards Birkhoff averages of continuous functions are constant on strongly unstable manifolds and forward Birkhoff averages of continuous functions are constant on stable manifolds, we obtain

$$\varphi^-(y) = \varphi^-(z) = \varphi^+(z) = \varphi^+(x).$$

Thus φ^- is constant on $N'(x)$ as desired. ∎

3.2. Proof of Theorem 4

Let $\mathrm{Diff}^r_\mu(M)$ be the space of C^r diffeomorphisms of M that preserve the smooth measure μ.

Lemma 1. There are a neighborhood $\mathscr{U}$ of f in $\mathrm{Diff}^1_\mu(M)$ and a constant $\alpha > 0$ with the following property. For any $g \in \mathscr{U}$ there is a subset $A_g \subset M$ with $\mu(A_g) > 0$ such that for every $x \in A_g$,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \|dg \mid E^c_g(g^j(x))\| \leqslant -\alpha.$$

Proof. Choose $\alpha > 0$ such that

$$\int_M \ln \|df \mid E^c_f(x)\| \, d\mu(x) < -\alpha. \tag{1}$$

Since the central bundle E^c_g depends continuously on the diffeomorphism g in the C^1 topology, there is a neighborhood $\mathscr{U} \subset \mathrm{Diff}^1_\mu(M)$ of f such that for any $g \in \mathscr{U}$,

$$\int_M \ln \|dg \mid E^c_g(x)\| \, d\mu(x) < -\alpha.$$

For $g \in \mathscr{U}$, let A_g be set of points x where the forward Birkhoff average of $\|dg \mid E^c_g(\cdot)\|$ is defined and less than $-\alpha$. It follows from the Birkhoff ergodic theorem (and the fact that μ is a probability measure) that $\mu(A_g) > 0$. If $x \in A_g$, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln \|dg \mid E^c_g(g^j(x))\| \leqslant -\alpha.$$

This completes the proof. ∎

Since

$$\frac{1}{n} \ln \|dg^n \mid E^c_g(x)\| \leqslant \frac{1}{n} \sum_{j=0}^{n-1} \ln \|dg \mid E^c_g(g^j(x))\|,$$

we see that g has negative central Lyapunov exponents on the set A_g. It now follows from Theorem 1 that the set A_g is open (mod 0) and $g \mid A_g$ has at most countably many ergodic components which are open (mod 0).

We proceed with the following result.

Lemma 2. There are a neighbourhood $\mathscr{V}$ of f in $\mathrm{Diff}^1_\mu(M)$ and a number $r_0 > 0$ such that, if $g \in \mathscr{V}$ is a C^2 diffeomorphism and $x \in A_g$ (the set defined in Lemma 1), then there is an $n \geqslant 0$ such that the size of the stable global manifold $W^{cs}(g^{-n}(x))$ is at least r_0.

Proof. Set $\sigma = \exp(-\alpha/3)$, where α is defined in (1). We call the number n a *σ-hyperbolic time for g at x* if

$$\|dg^j \mid E^c_g(g^{-n}(x))\| \leqslant \sigma^j,$$

for $0 \leqslant j \leqslant n$. Corollary 3.2 of ref. 21 and the remarks preceding it imply that if $g \in \mathscr{U}$ and $x \in A_g$, then there are infinitely many σ-hyperbolic times for g at x.

Denote by $B^{cs}(y, r)$ the ball in $V^{cs}(y)$ centered at y of radius r. It follows from Lemma 2.7 in ref. 21 that we can choose $r_0 > 0$ and a neighborhood $\mathscr{V} \subset \mathscr{U}$ of f such that for every C^2 diffeomorphism $g \in \mathscr{V}$ and any σ-hyperbolic time n for g at x,

$$d_n = \mathrm{diam}(g^j(B^{cs}(g^{-n}(x), r_0))) \leqslant \sigma^{j/2} \qquad \text{for} \quad 0 \leqslant j \leqslant n.$$

Since $\sigma < 1$, we can make d_n as small as we wish by choosing n to be a sufficiently large σ-hyperbolic time for g at x. In particular, we can ensure that $g^n(B^{cs}(g^{-n}(x), r_0))$ lies in the stable local manifold $V^{cs}(x)$. Then $B^{cs}(g^{-n}(x), r_0)$ is contained in the stable manifold $W^{cs}(g^{-n}(x))$ as claimed. ∎

We now repeat the proof of Theorem 1 for $g \mid A_g$. Instead of using $V^{cs}(x)$ to construct a neighborhood of x, we use $B^{cs}(g^{-n}(x), r_0)$ to construct a neighborhood of $g^{-n}(x)$, where n is a large σ-hyperbolic time for g at x. This gives us the following statement.

Lemma 3. There is a positive number $\tilde{r}_0 = \tilde{r}_0(f)$ such that any ergodic component of $g \mid A_g$ contains a ball of radius $\tilde{r}_0$.

It remains to show that the set A_g has full measure and $g|A_g$ is ergodic. Since f is accessible, the result from ref. 10 mentioned in the introduction tells us that the f-orbit of a.e. point in M is dense. Thus our desired claims hold when $g = f$, and they would hold for any g close enough to f if the accessibility property were open. This, however, is an open problem (see ref. 29 for some interesting results in this direction).

We will therefore exploit a weaker property, which is sufficient for our purpose (even in the case $g = f$). Given $\varepsilon > 0$, we say that a diffeomorphism g is *ε-accessible* if for every open ball B of radius ε the union of accessibility classes passing through B is M. An equivalent requirement is that the accessibility class of any point should enter every open ball of radius ε.

Lemma 4. Assume that f has the accessibility property. Then the following properties hold for every $\varepsilon > 0$.

(a) There exist $\ell > 0$ and $R > 0$ such that for any $p, q \in M$ one can find a *us*-path that starts at p, ends within distance $\varepsilon/2$ of q, and has at most ℓ legs, each of them with length at most R.

(b) There exists a neighborhood $\mathcal{U}$ of f in the space $\mathrm{Diff}^2(M)$ such that every $g \in \mathcal{U}$ is ε-accessible.

Proof. (a) Let $q_1, \ldots, q_N$ be an $\varepsilon/4$-net in M. For each $p \in M$ and each q_k, choose a *us*-path from p to q_k; let $\ell(p, k)$ be the number of legs and $R(p, k)$ the length of the longest leg in this path. Set

$$R(p) = \max_k R(p, k) \qquad \text{and} \qquad \ell(p) = \max_k \ell(p, k).$$

By continuity of the foliations W^u and W^s, every point $p \in M$ has a neighbourhood $U(p)$ such that, for each k, any point in $U(p)$ is joined to some point in $B(q_k, \varepsilon/4)$ by a *us*-path which has at most $\ell(p)$ legs each of length at most $2R(p)$. The sets $\{U(p)\}$ form an open cover of M. Let $\{U(p_1), \ldots, U(p_m)\}$ be a finite subcover. Then $R = \max_i 2R(p_i)$ and $\ell = \max_i \ell(p_i)$ satisfy the condition of part (a).

(b) The statement follows from (a) and the continuous dependence of the leaves of W^u and W^s on g. ∎

We proceed with the proof of the theorem. Given $\varepsilon > 0$, we say that an orbit $\mathrm{Orb}(x) = \{f^n(x) : n \in \mathbb{Z}\}$ is *ε-dense* if the points of the orbit form an ε-net.

Lemma 5. If g is ε-accessible, then almost every orbit is ε-dense.

Proof. It suffices to show that if B is an open ball of radius ε, then the orbit of, a.e., point enters B. To this end, let us call a point *good* if it has a neighborhood in which the orbit of a.e. point enters B. We now wish to show that an arbitrary point p is good. Since g is ε-accessible, there is a *us*-path $[z_0, \ldots, z_k]$ with $z_0 \in B$ and $z_k = p$. We shall show by induction on j that each point z_j is good.

This is obvious for $j = 0$.

Now suppose that z_j is good. Then z_j has a neighborhood N such that $\mathrm{Orb}(x) \cap B \neq \emptyset$ for a.e. $x \in N$. Let S be the subset of N consisting of points with this property that are also both forward and backward recurrent. It follows from the Poincaré recurrence theorem that S has full measure in N. If $x \in S$, any point $y \in W^s(x) \cup W^u(x)$ has the property that $\mathrm{Orb}(y) \cap B \neq \emptyset$. The absolute continuity of the foliations W^s and W^u means that the set

$$\bigcup_{x \in S} W^s(x) \cup W^u(x)$$

has full measure in the set

$$\bigcup_{x \in N} W^s(x) \cup W^u(x).$$

The latter is a neighborhood of z_{j+1}. Hence z_{j+1} is good. ▮

Theorem 4 now follows from Lemmas 3, 4 and 5. ▮

3.3. Proof of Theorem 2

If f is essentially accessible, the accessibility class of a.e. point is dense in M. With minor modifications, the proof of Lemma 5 shows that this property imples that a.e. point has a dense orbit. Theorem 2 is then immediate from Theorem 1. ▮

3.4. Proof of Theorem 3

By Theorem 2, f is ergodic and has negative central exponents, a.e., with respect to μ. Hence there exists $\beta > 0$ such that for, a.e., $x \in M$

$$\lim_{n \to +\infty} \frac{1}{n} \ln \|df^n \mid E^c_f(x)\| \leqslant -\beta.$$

Integrating over M we obtain

$$\lim_{n \to \infty} \frac{1}{n} \int_M \ln \|df^n \mid E^c_f(x)\| \, d\mu(x) \leqslant -\beta.$$

In particular, there exists $n_0 > 0$ such that

$$\int_M \ln \|df^{n_0} \mid E^c_f(x)\| \, d\mu(x) < 0.$$

Hence f^{n_0} satisfies the hypotheses of Theorem 4 and thus is stably ergodic. It follows that f itself is stably ergodic. ▮

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